UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M12B: Algebra 2

COURSE CODE	:	**MATHM12B**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1038-3-210

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give the definition of a *group*, defining the terms you use. Prove that in any group the identity element is unique, and each element has a unique inverse.

   (b) Determine whether or not the following sets G under the given operation $\star$ are groups, justifying your answer:

   (i) $G = \mathbb{R} - \{-2\}$, $a \star b = ab + 2a + 2b + 2$,

   (ii) $G = \mathbb{R}$, $a \star b = a - b$,

   (iii) $G = \{x \in \mathbb{R} : x \geq 0\}$, $a \star b = +\sqrt{a^2 + b^2}$.

2. (a) State (do not prove) Lagrange's Theorem. Hence prove that in a finite group G the order of any element divides the order of the group.

   (b) Deduce that for any prime p if $a \not\equiv 0 \pmod{p}$ then $a^{p-1} \equiv 1 \pmod{p}$.

   (c) Find $2^{3599} \pmod{37}$.

   (d) Find an number x such that $x^7 \equiv 2 \pmod{37}$

3. (a) Stating clearly any results you use, prove that for any two $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

   (b) Find $\det \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$, expressing your answer as a product of linear factors.

4. (a) Let A be an $n \times n$ matrix over $\mathbb{R}$. Give the definition of:

(i) an *eigenvalue* λ of A;

(ii) an *eigenvector* corresponding to λ;

(iii) the *eigenspace* E_λ of A;

(iv) A is *diagonalizable* (over $\mathbb{R}$). State (do not prove) the basic criterion for a matrix to be diagonalisable.

(b) Let λ_i $(i = 1, .., r)$ be the distinct eigenvalues of A; prove that the sum $\sum_{i=1}^{r} E_{\lambda_i}$ is direct. Deduce that if $\sum_{i=1}^{r} \dim(E_{\lambda_i}) = n$ then A is diagonalisable.

(c) Prove that the following matrix is diagonalisable:

$$\begin{pmatrix} 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & 0 & -2 & -2 \end{pmatrix}$$

5. Let $A = \begin{pmatrix} 3/2 & -1 \\ 1/2 & 0 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find A^n for $n \in \mathbf{N}$

(iii) Solve the system of difference equations

$$\begin{aligned} x_{n+1} &= \tfrac{3}{2}x_n - y_n \\ y_{n+1} &= \tfrac{1}{2}x_n \end{aligned}$$

given that $x_0 = 0$, $y_0 = 1$.

Find the limit, as $n \longrightarrow \infty$ of x_n.

6. (a) Let A be a real symmetric matrix and let $\mathbf{u}$, $\mathbf{v}$ be eigenvectors associated to the (real) eigenvalues λ and μ respectively, where $\lambda \neq \mu$. Prove that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors.

(b) Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

END OF PAPER